FLUID MECHANICS ORIGINAL PAPER

Dimensional analysis in the venous system

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Abstract Dimensional analysis, a standard method of Fluid Mechanics, was applied to the field of venous hemodynamics. Three independent physical quantities, velocity, length and pressure, were chosen and seven other ones were used to derive the non-dimensional terms. The mathematical burden was reduced to the minimum and the attention was focused on the results. Among them, a new formulation of an already known non-dimensional term, recalled the flow-length (FL), was identified and selected for a deeper experimental study.

Keywords Buckingham Pi theorem, Fluid mechanics, Venous hemodynamics, Non-dimensional numbers, Flow-length.

Introduction and aim

The paper resumes the content of a Masterclass held recently in an international phlebology meeting[1]. Dimensional analysis (DA) is a standard method in Physics and Engineering, especially in Fluid Mechanics[2,3,4], and it is used to clarify, to check and to foresee the mathematical relationship between physical quantities. As a practical consequence, non-dimensional terms detected by DA are used in modelling, when comparing a prototype to a different size scaled model, i.e. bigger or smaller.

DA can be practically used:
- 1) trivially to **remind** the structure of well-known physical laws;
- 2) to **check** the consistency of a new theoretical relationship;
- 3) to search for **new possible laws**
  - a) in a **brute force** extensive analysis;
  - b) in a **selected set** of quantities under study.

The results can be:
- 1) **trivial**, i.e. clear and simply obtained by inspection;
- 2) **already known**, i.e. a different way to represent well-assessed laws;
- 3) new previously unexplored connections
  - a) **with** a clear physical interpretation;
  - b) **without** a ready physical interpretation.

The main procedure in DA is the Π_i theorem or Buckingham theorem [5-10], which provides practical hints for the computation of useful non-dimensional terms. Each term is a non-dimensional product Π_i of physical quantities, their sum being non-dimensional too. Therefore, the words "terms" and "products" are used in this paper in an interchangeable way.

The aim of this paper was to apply the Π_i theorem to the field of venous hemodynamics, with the strong hope to find a practical relationship between easy measurable physical quantities, collected during a widely available diagnostic investigation. Venous hemodynamics was never extensively studied with DA. Its theoretical results will be then investigated in future experimental works, in order to capture the coefficients of the terms, applied to the venous system.

Main physical entities

#	dimensions	name	symbol
1	$[L][T]^{-1}$	Velocity	V
2	$[L]$	Length	D
3	$[M][L]^{-1}[T]^{-2}$	Pressure	P

Table I - The selected main physical entities, their name, symbol and dimensions, pertaining to the current dimensional analysis. They allowed the formulation of a starting core of equations, which was then minimally changed to derive each Π_i term.

Method

The international system of units (SI, from the French "Système International d'unités") was chosen[11, i].

The next step consisted then in writing the list (Table I and II) of the potentially involved N physical quantities, together with their dimensions.

An arbitrarily choice of Q main quantities from the previous list of N was performed, provided they contained all the Q fundamental physical quantities of the chosen measuring system (e.g. M, L and T in SI and Q=3)[ii] .

N=10 physical quantities were selected[iii], choosing as the main group Q=3 quantities, the velocity V, the length D and the pressure P, their dimensions being shown in Table I.

Their independence was checked, showing that except for the trivial solution (0,0,0) no combination of their exponent values (a,b,c) existed[12], to provide a non-dimensional product[iv].

Indeed, raising the three main physical quantities to the unknown exponents a b c

$[V]^a\ [D]^b\ [P]^c =$

$= [L]^a\ [T]^{-a}\ [L]^b\ [M]^c\ [L]^{-c}\ [T]^{-2c} =$

$= [M]^c\ [L]^{a+b-c}\ [T]^{-a-2c}$

and equating to 0 each exponent in the SI base units, the following system of equations could be laid down

$0 = a+b-c$; (for [L])

$0 = c$; (for [M])

$0 = -a -2c$; (for [T])

The unique solution of the system was a = b = c = 0, i.e. the three chosen quantities were independent on each other and could constitute a base of the measuring system. The above described procedure provided a core equation set for all the other subsequent computation.

Any of the examples shown in Table III could be chosen to represent practically the corresponding main quantities, i.e. velocity, length and pressure. For instance, the **diastolic velocity**, the **calibre** and the **intravascular pressure** in any vein could constitute a valid triad and were effectively used in almost all practical implementations of the method.

A quicker notation for dimensions, omitting the square brackets, was adopted in the text below. **N-Q=7(10-3) Π_i products** (Table II) were found and for each of them the equations were written and solved.

The computing procedure essentially consisted in taking each representative quantity from table II and adding it separately to the main quantities $V^a\ D^b\ P^c$. For each additional quantity, a new computational procedure was performed, producing new coefficients for a modified core of linear equations.

The system was then solved, following in each case the same computational scheme. Finally, the computed coefficients were applied to get each corresponding non-dimensional term[13,14].

Π_i terms

Π	dimensions	name	symbol	Π formula	Interpretation
Π_1	$[T]$	time	t	$VD^{-1}t$	Stokes number, FL
Π_2	$[M][L]^{-1}[T]^{-1}$	absolute viscosity	μ	$VD^{-1}P^{-1}\mu$	Reynolds number
Π_3	$[L]^3[T]^{-1}$	flow	Φ	$V^{-1}D^{-2}\Phi$	Leonardo continuity law
Π_4	$[M][L]^{-3}$	density	ρ	$V^2 P^{-1}\rho$	Bernouilli kinetic term
Π_5	$[L][T]^{-2}$	acceleration	ac	$V^{-2}D\ ac$	Froude number
Π_6	$[M][T]^{-2}$	superficial tension	σ	$D^{-1}P^{-1}\sigma$	Young-Laplace law
Π_7	$[T]^{-1}$	angular velocity	ω	$V^{-1}D\omega$	Strouhal number

Table II - The Π_i terms of the current dimensional analysis, each one derived by adding a physical quantity (name) to the set of the main physical quantities (velocity, length, pressure). A brief interpretation is given in the right column. For a detailed description see the text.

Results

Trivial results

A trivial immediate non-dimensional result is given by the ratio between two physical quantities of the same dimension, both taken from the same example group. For instance, transmural pressure divided by the intravascular pressure is non-dimensional. The same applies separately to length and velocity, thus having for instance the non-dimensional ratios given by the vein diameter (D) divided by the height at the groin (H) (D/H) and the mean velocity ratio between the great saphenous vein (GSV) and the common femoral vein (CFV) (V_{GSV}/V_{CFV}). In addition, the same property holds for the powers of the main physical entities, for instance:

- for the surface ($[L]^2$), the GSV terminal valve area (A_{TV}) divided by the GSV cross-section (A_{GSV}) at 2 cm from the junction (A_{TV}/A_{GSV});
- for the volume ($[L]^3$) , the ratio between the blood volume ejected by the calf pump (EV) and the volume of the lower limb (LV), computed by the frustum method[15] (EV/LV).

These obvious results were not treated extensively, because they can be derived by simple inspection and do not require any computation.

Examples of main physical quantities in the venous system

Velocity	Max or mean, systolic or **diastolic velocity** in any vein and anatomical site
Length	Max or min **calibre** of a vessel, height of the body or at the Lewis angle or at the groin, vein wall thickness, length of a venous shunt, etc.
Pressure	Systolic or diastolic pressure in the aortic arch, difference of pressure, transmural pressure, **intravascular pressure**

Table III - Physical instances of the main physical quantities in veins. Quantities marked in bold were practically used for the real computation of the non-dimensional terms.

Dimensional analysis

The text below describes the computation details for each Π_i term. The computation of the first term was outlined in a detailed and plain way, while for all the other terms the details were omitted.

Time

The term Π_1 was computed adding the time (**t**) quantity (1st line in Table II) to the core equations. Using for the dimensions the unknown coefficients **a, b, c** the following relationship could be laid down

$$V^a D^b P^c t = M^c L^{a+b-c} T^{-a-2c} T$$

leading then to the algebraic equation system

$0 = a+b-c$ (for [L])

$0 = c$ (for [M])

$0 = -a-2c+1$ (for [T])

and solving **$a = 1; b = -1; c = 0$**.

Finally, substituting with the computed values, the non-dimensional term **$VD^{-1} t$** was obtained.

Viscosity

The term Π_2 was computed adding the viscosity (μ) quantity (2nd line in Table II) to the core equations.

$$V^a D^b P^c \mu = M^c L^{a+b-c} T^{-a-2c} ML^{-1} T^{-1}$$

and then

$0 = a+b-c-1$ (for [L])

$0 = c+1$ (for [M])

$0 = -a-2c-1$ (for [T])

and solving **$a = 1; b = -1; c = -1$**.

Finally, the non-dimensional term **$V D^{-1} P^{-1}\mu$** was obtained.

Flow

The term Π_3 was computed adding the flow (Φ) quantity (3rd line in Table II) to the core equations.

$$V^a D^b P^c \Phi = M^c L^{a+b-c} T^{-a-2c} L^3 T^{-1}$$

and then

$0 = a+b-c+3$ (for [L])

$0 = c$ (for [M])

$0 = -a-2c-1$ (for [T])

and solving **$a = -1; b = -2; c = 0$**.

Finally, the non-dimensional term **$V^{-1}D^{-2}\Phi$** was obtained.

Density

The term Π_4 was computed adding the density (ρ) quantity (4th line in Table II) to the core equations.

$$V^a D^b P^c \rho = M^c L^{a+b-c} T^{-a-2c} ML^{-3}$$

and then

$0 = a+b-c-3$ (for [L])

$0 = c+1$ (for [M])

$0 = -a-2c$ (for [T])

and solving **$a = 2; b = 0; c = -1$**.

Finally, the non-dimensional term **$V^2P^{-1}\rho$** was obtained.

Acceleration

The term Π_5 was computed adding the acceleration (**ac**) quantity (5th line in Table II) to the core equations.

$$V^a D^b P^c ac = M^c L^{a+b-c} T^{-a-2c} LT^{-2}$$

and then

$0 = a+b-c+1$ (for [L])

$0 = c$ (for [M])

$0 = -a-2c-2$ (for [T])

and solving **$a = -2; b = 1; c = 0$**.

Finally, the non-dimensional term **$V^{-2}D\ ac$** was obtained.

Superficial tension

The term Π_6 was computed adding the superficial tension (σ) quantity (6th line in Table II) to the core equations.

$$V^a D^b P^c \sigma = M^c L^{a+b-c} T^{-a-2c} MT^{-2}$$

and then

$0 = a+b-c$ (for [L])

0 = c+1 (for [M])

0 = -a-2c-2 (for [T])

and solving **a = 0; b = -1; c = -1**.

Finally, the non-dimensional term $\mathbf{D^{-1}P^{-1}}\sigma$ was obtained.

Angular velocity

The term Π_7 was computed adding the angular velocity (ω) quantity (7th line in Table II) to the core equations.

$$\mathbf{V^a\, D^b\, P^c \omega = M^c\, L^{a+b-c}\, T^{-a-2c}\, T^{-1}}$$

and then

0 = a+b-c (for [L])

0 = c (for [M])

0 = -a-2c-1 (for [T])

and solving **a = -1; b = 1; c = 0**.

Finally, the non-dimensional term $\mathbf{V^{-1}D}\ \omega$ was obtained.

Discussion

Seven non-dimensional terms were obtained from the performed dimensional analysis. All these terms correspond to well-known non-dimensional products already in use in fluid mechanics (Table II and IV), thus no completely new relationship was detected. However, several terms were practically applied with different quantities other than those usually adopted, as detailed case by case below.

The following section requires a simultaneous consultation of Table IV.

The Π_1 term resembles the well-known Stokes number, which is generally applied in multiphase systems and to streamlines along suspended particles in a fluid. Although blood is a multiphase system, its particles (red blood cells) are very small, compared to the venous caliber, though more similar instead in capillaries. Using quantities which rely better to daily venous investigations appears very promising.

For instance, using the time length of the forward flow/venous reflux and the mean velocity and the venous diameter, the quantity Vt was interpreted as the covered distance or the mean depth of flow/reflux. Therefore, the Π_1 term was considered as a new formulation of the Stokes number and assigned the acronym **FL (Flow-Length)**, i.e. the mean depth defined above using the vein diameter as unit[16] (Fig 1).

Main non-dimensional numbers in fluid mechanics

Name	Formula	Main phenomenon
Stokes (Stk)	$Stk=\tau V/D$	Dispersion
Reynolds (Re)	$Re=VD\rho/\mu$	Viscous
Froude (Fr)	$Fr=V/(gD)^{1/2}$	Gravity
Euler (Eu)	$Eu=\Delta p/\rho V^2$	Pressure
Weber (We)	$We=\rho V^2 D/\sigma$	Surface
Womersley (α)	$\alpha=R(\omega R/\mu)^{1/2}$	Pulsatility
Strouhal (St)	$St=\omega D/V$	Vortices

Table IV - The most used non-dimensional numbers in modelling. Symbols: τ relaxation time, V velocity, D length, ρ density, μ viscosity, g gravitational acceleration, p pressure, σ superficial tension, R radius, ω angular velocity. For additional information see the text.

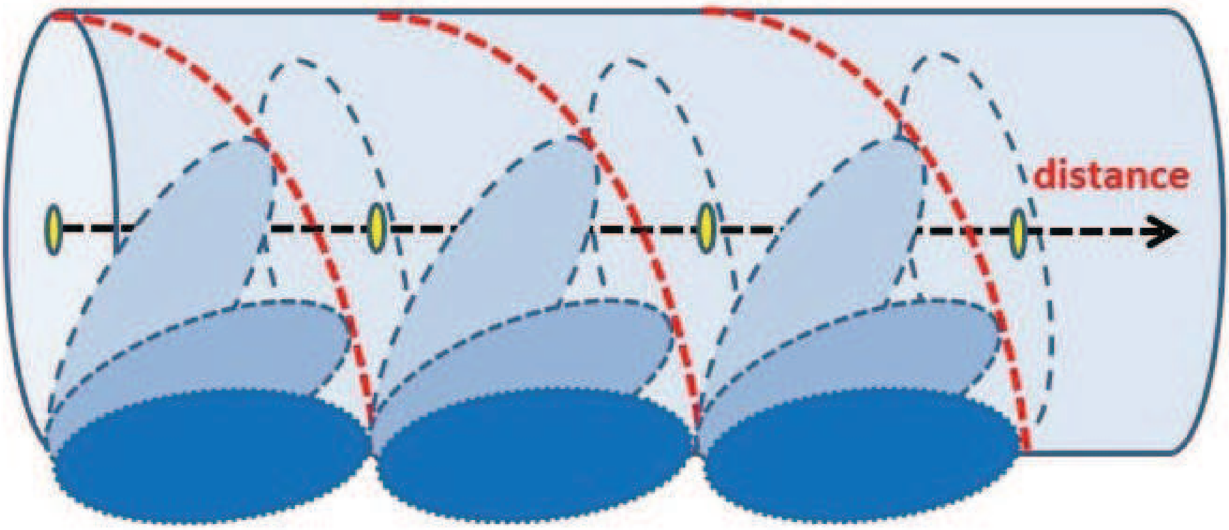

*Figure 1 - Using the diameter as Length Unit. The flow-length (FL) counts how many diameters are covered by the fluid in the time length of flow. Imagine the cross-section like a rolling coin, covering the distance given by the mean velocity **V** multiplied by the time length **t** of flow.*

FL counts indeed how many diameters are covered by the flow, in any pathophysiologic conditions, like the spontaneous breath for forward flow or the Valsalva for the reflux. FL was selected for a deeper experimental study, as it can be measured in a very simple way by echocolordoppler, so that it could be part of any vascular ultrasound investigation and report. Being applied to a different context, FL is therefore a reformulation of the Stokes formula for venous (or arterial) macro-circulation.

As regards the time length, FL can be easily applied when there is a well recognizable cycle (and thus an identifiable time), while for a continuous flow with a steady velocity value the Froude number seems a better choice.

The Π_2 term $\mathbf{VD^{-1}\ P^{-1}}\mu$ has a non-dimensional property which holds also for its reciprocal $\mathbf{P/V} * \mathbf{D}/\mu$, where P is the pressure, V the mean velocity, D the diameter and μ the absolute viscosity. As the dimension of the energy density is $[\rho V^2] = [P]$ and consequently $[\rho V] = [P/V]$, where ρ is the density, substituting in the formula for the reciprocal of the Π_2 term the following relationship is obtained

$$\mathbf{P/V} * D/\mu = \rho\mathbf{V} * D/\mu = Re$$

where Re is the Reynolds number.

Thus, the Π_2 term is the reciprocal of the Reynolds number, which is commonly used in fluid mechanics. In the same way Re needs the value of blood viscosity, the Π_2 term is of difficult computation too, being the intravascular pressure generally unknown.

The Π_3 term $\mathbf{V^{-1}\ D^{-2}}\Phi$ could be related to the well-known Leonardo law (equation of continuity), while the Π_4 term $\mathbf{V^2\ P^{-1}}\rho$ could be referred to the kinetic term of the Bernouilli equation ($\mathbf{P} = \rho\mathbf{V^2}$).

The Π_5 term $\mathbf{V^{-2}\ D\ ac}$ is the reciprocal of the squared Froude number (Fr). Generally Fr is applied to open channels, where the symbol g in the formula is the gravity acceleration constant. In venous hemodynamics the interest is in pipes, where the flow acceleration can be used. As all the quantities are easily measurable in a common echocolordoppler investigation, Fr can be computed too in an easy way.

The Π_6 term $\mathbf{D^{-1}\ P^{-1}}\sigma$ reflects the Young-Laplace formula $\mathbf{P} = \sigma/\mathbf{D}$, experimentally discovered by Young in 1804 and then reformulated theoretically one year later by Laplace[17] (Fung 1981, p.19). Alternatively, substituting P with the kinetic term of the Bernouilli equation, it becomes the reciprocal of the Weber number (We).

The Π_7 term $\mathbf{V^{-1} D} \omega$ is the Strouhal number (St), where ω is the pulsatility of flow (like the heart rate) and deals with the generation of vortices downstream of an obstacle to flow, as in a pulsatile flow in a carotid stenosis[18] (Fung 1984, p.46). In the Π_7 term instead, it is possible to consider ω as the angular velocity of flow (there is an additional angular dimension), inside the cross section of the pipe. This interpretation involves flow velocities in other directions than the commonly used axial ones and suggests a deeper theoretical investigation.

Conclusion

DA is an easy method to get interesting theoretically relationship, without any experimental data collection. Apart from the huge work in describing the theoretical method and the ordered results, the practical application of DA for the current paper required almost 1 hour of work, sitting down at a desk and computing all the values manually. Using a program would have substantially reduced the computation time to a few minutes. The collected results cover a great part of fluid mechanics of the past 150 years, undoubtedly showing the power of the method.

However, DA provides reliable formulas apart from the non-dimensional coefficients, which require instead a separate experimental investigation, in order to assess their value and behaviour.

Accordingly, an experimental work to assess the reliability and the clinical meaning of FL (or the modified Stokes non-dimensional number) is currently underway in chronic venous diseases[16], comparing FL to the Froude and Reynolds numbers.

In the same way, interesting non-dimensional relationship like the Π_{6-7} terms were considered for a deeper theoretical approach, in order to provide an application more near to the venous system as well as to draw a future experimental design. Indeed, the main aim of the current paper was to find an useful relationship, adopting typical measurements of venous hemodynamics. Finally, the results show the possibility in the human of a non-invasive measurement by a simple echocolordoppler of most fluid mechanics non-dimensional numbers, generally considered just a theoretical and academic subject, thus never clinically reported.

Endnotes

[i] A reduced set of the SI units Mass [M] Kilogram (Kg), Length [L] meter (m) and Time [T] second (s) was used, as the remaining SI units (Ampère, Candela, Kelvin, Mole) were considered of minor importance in venous hemodynamics.

[ii] DA does not depend on the choice of the main physical quantities, thus just the quantities which fit better to computation or measurement were used

[iii] Although the choice was arbitrary, the biophysical and clinical experience was a useful help in the selection of these physical quantities.

[iv] Intentionally, the matrix notation was not used in this paper. However, it's worth noting that the rank of the dimensional matrix of all the considered physical quantities is M=3.

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